3d Gauge Theories, Symplectic Duality and Knot Homology II

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What does it mean for a QFT to be Lagrangian?

In a QFT, there are operators (e.g. local operators) O_A, and there are prescriptions for calculating correlation functions of products of operators

$$\langle O_{A_1} \dots O_{A_n} \rangle. \tag{1}$$

If the operators are local, this tells you something about scattering processes. If the QFT is Lagrangian, then there is the additional data of

1. a set of fields ϕ,
2. a Lagrangian density $\mathcal{L}(\phi, \partial\phi, \dots)$, and
3. a map from operators to functionals on fields (not entirely correct) sending an operator O_A to a functional $F_A(\phi)$

satisfying the condition that correlation functions can be computed as path integrals

$$\langle O_{A_1} \dots O_{A_n} \rangle = \frac{\int D\phi e^{i\int \mathcal{L}} F_{A_1} \dots F_{A_n}}{\int D\phi e^{i\int \mathcal{L}}}. \tag{2}$$

More generally operators may be mapped to functionals on fields with boundary conditions; these are called disorder operators.

If the theory is free, meaning that $\mathcal{L}$ is quadratic and the equations of motion are linear and decoupled, it is possible to explicitly compute the path integral. If the theory is weakly coupled, meaning that $\mathcal{L}$ is quadratic plus a small correction, then we can hope to compute the path integral using perturbation theory (Feynman diagrams).

If the theory is strongly coupled, meaning that $\mathcal{L}$ is quadratic plus a large correction, all bets are off; there's no reason that perturbation theory should work. One might nevertheless hope to gain some intuitions by formally manipulating the path integral.

The strength of coupling depends on an energy scale when they have dimensions.

Q: are there examples where the coupling constants are dimensionless?

A: yes, most notably in QCD. Classically the gauge coupling is dimensionless. In the quantum setting there are corrections that give it a dimension.

Recall the setting from the first talk. Let

$$\pi : M \to M_0 \tag{3}$$

be a conical symplectic resolution, so M_0 is an affine cone. We will want M and M_0 to be hyperkähler, so there is a hyperk"ahler form ω and a complex symplectic form $\Omega = \omega_1 + i\omega_2$. There is an action of $U(1)_\epsilon$ on M such that Ω has weight 2 extending to an action of $\mathbb{C}^\times_\epsilon$ on M_0. Let G_M be the group of hyperk"ahler isometries of M commuting with $U(1)_\epsilon$; then we also want an action of $U(1)_\xi \subseteq G_M$ wiht isolated fixed points.

This set up arises in 3d $N = 4$ SYM as follows. We will work on $\mathbb{R}^{2,1}$. Its oriented isometry group is the Poincaré group $\mathbb{R}^{2,1} \rtimes \mathrm{SO}(2,1)$. The N-extended SUSY algebra is an odd Lie superalgebra extending the Poincaré Lie algebra with N odd generators $Q^A_\alpha, A = 1, \ldots N, \alpha = 1, 2$ denoting components of a spinor representation of $SO(2,1)$, or equivalently a fundamental representation of $\mathrm{SL}_2(\mathbb{R})$. In this case, the anticommutator relations are

$$\{Q^A_\alpha, Q^B_\beta\} = \delta^{AB} \sigma^\mu_{\alpha\beta} P_\mu \tag{4}$$

where P_μ generate the translations in $\mathbb{R}^{2,1}$ and σ are the matrices

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma^1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{5}$$

This super Lie algebra has a central extension in which the anticommutator becomes

$$\{Q^A_\alpha, Q^B_\beta\} = \delta^{AB} \sigma^\mu_{\alpha\beta} P_\mu + Z^{AB} \epsilon_{\alpha\beta}. \tag{6}$$

Note that there is an extra $\mathrm{SO}(N)_R$ (the R is an index here) symmetry acting on the odd / SUSY generators. This is called the R-symmetry group. When $N = 4$ we have

$$\mathrm{Spin}(4)_R \cong \mathrm{SU}(2)_C \times SU(2)_H \tag{7}$$

where C, H are indices standing for Coulomb and Higgs respectively. With respect to this symmetry Z^{AB} decomposes into two components.

3d $N = 4$ SYM is a dimensional reduction of a 4d $N = 2$ theory (a Class S theory).

The fields of this theory come in collections related by the SUSY generators called multiplets. There is a hypermultiplet of two complex fields $(X(x), Y(x))$ (or four real fields) and two complex fermions, which are mixed up by supersymmetry. $(X, Y^\dagger)$ lives in the defining / vector representation of $\mathrm{SU}(2)_H$.

On $\mathbb{R}^{2,1}$, the space of fields, meaning the space of values of fields at a point, or equivalently the space of constant field configurations, must be hyperkähler. In this case it is $\mathbb{C}^2 \cong T^*(\mathbb{C})$ with its standard hyperkähler structure

$$ds^2 = |dX|^2 + |dY|^2, \omega = idX \wedge d\overline{X} + idY \wedge d\overline{Y}, \Omega = dX \wedge dY. \tag{8}$$

$U(1)_H \subsetneq \mathrm{SU}(2)_H$ preserves the complex structure and rotates X, Y, Ω with weights $1, 1, 2$.

Then there is a vector multiplet. To describe it we choose a compact Lie group G, and then the bosons in the vector multiplet consist of a G-connection (gauge field) A_μ, a real scalar field σ valued in $\mathfrak{g}$, and a complex scalar field φ valued in $\mathfrak{g}_\mathbb{C}$. We will think of A_μ as a $\mathfrak{g}$-valued 1-form. (σ, φ) lives in the defining / vector representation of $\mathrm{SO}(3)_H$.

The space of fields is again hyperkähler. This is easiest to see when $G = U(1)$. Then $dA = *d\gamma, \gamma \in S^1$, and the space of fields $(\sigma, \varphi, \gamma)$ turns out to be $\mathbb{R}^3 \times S^1$ with hyperkähler structure

$$ds^2 = |d(\sigma + i\gamma)|^2 + |d\varphi|^2, \Omega = d(\sigma + i\gamma) \wedge d\varphi. \tag{9}$$

Q: what's the relevance of these theories to Theory X?

A: they are not always compactifications of Theory X, for which I apologize. Some of them arise in this way, as compactifications $X[\Sigma \times S^1]$ if $X[\Sigma]$ is a Class S theory with a description as a Lagrangian gauge theory.

In general, let G be a compact Lie group and let $V = (T^*\mathbb{C})^N$ with its hyperkähler structure. Our fields are hypermultiplets $(X_i, Y_i)_{i=1}^N$ and vector multiplets (A_μ, σ, φ). There are three types of parameters. First, there is a parameter g_a for each simple factor G_a of G so that g_a^2 has units mass.

Let G_H be a group of isometries acting on V, independently of G. Let G_C be a group of isometries acting on the bosons in the vector multiplet, independently of G. Classically this is the product of the $U(1)$ factors of G acting on γ (associated to A_μ) by rotation, but quantum mechanically it can be nonabelian.

Second, there are mass parameters

$$m \in \mathfrak{h}_{G_H} \otimes \mathfrak{su}(2)_C \tag{10}$$

and FI parameters

$$t \in \mathfrak{h}_{G_C} \otimes \mathfrak{su}(2)_H. \tag{11}$$

(Some discussion about frozen multiplets happened here that I didn't quite catch.)

The vacuum / BPS equations are classically $dW = dh = 0$ modulo the action of the gauge group G. The splitting between W and h only arises here because we chose complex structures above. Here W is a complex superpotential

$$W = \varphi \cdot \mu_G^{\mathbb{C}} + m^{\mathbb{C}} \cdot \mu_{G_H}^{\mathbb{C}} + t^{\mathbb{C}} \cdot \varphi \tag{12}$$

where $\mu_G^{\mathbb{C}}$ denotes a complex moment map for G-symmetry and φ is a complex moment map for G_C-symmetry. h is the real version

$$h = \sigma \cdot \mu_G + m \cdot \mu_{G_H} + t \cdot \sigma. \tag{13}$$

If the m and t parameters are set to zero, the classical moduli space (of solutions to the above equations modulo the action of G) has at least two components. One of them is the Higgs branch

$$M_H^0 = \{\sigma = \varphi = \mu_G^{\mathbb{C}} = \mu_G^{\mathbb{R}} = 0\}/G = V///G \tag{14}$$

where $///$ denotes a hyperkähler quotient, and one is the Coulomb branch

$$M_C^0 = \{X = Y = 0\} = (\mathbb{R}^3 \times S^1)^{\mathrm{rank}(G)}/W_G \tag{15}$$

where W_G is the Weyl group and the identification comes from looking at the eigenvalues of φ and σ (which can be simultaneously diagonalized) and γ.

Non-classically, the Coulomb branch acquires quantum corrections which change its topology and hyperkähler structure.

In symplectic duality there are mirror pairs of 3d theories whose Higgs and Coulomb branches swap.

(Some questions from the audience happened here that I didn't catch.)